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POLE ASSIGNMENT IN VIBRATORY SYSTEMS BY MULTI-INPUT CONTROL

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The problem of reassigning some poles of a vibratory system, while keeping the other poles unchanged, is considered. The problem may be solved uniquely by single-input state feedback control. A family of solutions to the partial pole assignment problem may be obtained by applying multi-input control forces. An algorithm for determining a multi-input control which is small in some sense is presented. The non-iterative algorithm proposed, defines a closed-form solution to the partial pole assignment problem in its natural second order form, and no first order realization is used. The reduction in the control effort achieved by the proposed method is demonstrated by numerical examples.

1. INTRODUCTION

The free vibrations of a linear, time-invariant, vibratory system are governed by the second order system

$$\mathbf{M}\ddot{\mathbf{z}} + \mathbf{C}\dot{\mathbf{z}} + \mathbf{K}\mathbf{z} = \mathbf{0}, \tag{1}$$

where $\mathbf{M}, \mathbf{C}, \mathbf{K} \in \Re^{n \times n}$ are symmetric matrices, $\mathbf{M}$ is positive definite, $\mathbf{C}$ and $\mathbf{K}$ are positive semi-definite, $\mathbf{z} \equiv \mathbf{z}(t) \in \Re^n$, and dots indicate derivatives with respect to the time t. Using separation of variables

$$\mathbf{z}(t) = \mathbf{x}e^{\lambda t}, \tag{2}$$

where $\mathbf{x}$ is a constant vector, leads to the eigenvalue problem

$$(\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K})\mathbf{x} = \mathbf{0}. \tag{3}$$

The characteristic polynomial associated with the *open-loop* system (1) is defined by

$$P_o(\lambda) = \det(\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K}). \tag{4}$$

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Then, clearly $P_o(\lambda)$ is a polynomial in λ of degree $2n$. Its $2n$ roots $\{\lambda_i\}_{i=1}^{2n}$ which satisfy

$$P_o(\lambda_i) = 0 \tag{5}$$

are the poles of the system. Associated with each pole λ_i, $i = 1, 2, \dots, 2n$, there exists a mode shape, or eigenvector, $\mathbf{x}_i$ which non-trivially solves

$$(\lambda_i^2\mathbf{M} + \lambda_i\mathbf{C} + \mathbf{K})\mathbf{x_i} = \mathbf{0}. \tag{6}$$

Let the spectral matrix be

$$\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \dots, \lambda_{2n}\}, \tag{7}$$

and the modal matrix

$$\mathbf{X} = [\mathbf{x}_1 | \mathbf{x}_2 | \dots | \mathbf{x}_{2n}] \in \Re^{n\times n}. \tag{8}$$

Then the response of the system to the initial conditions

$$\mathbf{z}(0) = \mathbf{z}_0, \quad \dot{\mathbf{z}}(0) = \mathbf{v}_0 \tag{9}$$

is given by

$$\mathbf{z}(t) = \sum_{i=1}^{2n} a_i \mathbf{x}_i \mathrm{e}^{\lambda_i t}, \tag{10}$$

where $\mathbf{a} = (a_1, a_2, \dots, a_{2n})^{\mathrm{T}}$ is the solution of

$$\begin{pmatrix} \mathbf{z}_0 \\ \mathbf{v}_0 \end{pmatrix} = \begin{bmatrix} \mathbf{X} \\ \mathbf{X}\Lambda \end{bmatrix} \mathbf{a}. \tag{11}$$

It is well known (see e.g. reference [1] that, under the conditions that we have imposed on the system, $\mathbf{Re}(\lambda_i) \leqslant 0$ for all $i = 1, 2, \dots, 2n$. Hence, the response (10) of equation (1) is bounded for arbitrary initial conditions $\mathbf{z}_0$ and $\mathbf{v}_0$. The response of the system to the initial conditions is required in some applications to diminish rapidly. This objective can be achieved by relocating some poles of the system in the complex plane. Suppose we wish to alter the location of the poles by applying the control force $\mathbf{b}u(t)$, where $\mathbf{b}$ a constant vector, to the system. Then the dynamics of the controlled system is governed by

$$\mathbf{M}\ddot{\mathbf{z}} + \mathbf{C}\dot{\mathbf{z}} + \mathbf{K}\mathbf{z} = \mathbf{b}u(t). \tag{12}$$

The locations of the poles will be independent of the time parameter if we choose state feedback, i.e.

$$u(t) = \mathbf{f}^{\mathrm{T}}\dot{\mathbf{z}} + \mathbf{g}^{\mathrm{T}}\mathbf{z}, \qquad \mathbf{f}, \mathbf{g} \in \Re^n, \tag{13}$$

since equation (12) may be written in the form

$$\mathbf{M}\ddot{\mathbf{z}} + \hat{\mathbf{C}}\dot{\mathbf{z}} + \hat{\mathbf{K}}\mathbf{z} = \mathbf{0}, \tag{14}$$

where

$$\hat{\mathbf{C}} = \mathbf{C} - \mathbf{b}\mathbf{f}^{\mathrm{T}} \quad \text{and} \quad \hat{\mathbf{K}} = \mathbf{K} - \mathbf{b}\mathbf{g}^{\mathrm{T}}. \tag{15}$$

Separation of variables,

$$\mathbf{z}(t) = \mathbf{y}e^{\mu t}, \tag{16}$$

$\mathbf{y}$ a constant vector and μ scalar, leads to the *closed-loop* eigenvalue problem

$$(\mu^2\mathbf{M} + \mu\hat{\mathbf{C}} + \hat{\mathbf{K}})\mathbf{y} = \mathbf{0}. \tag{17}$$

We define the characteristic polynomial of the closed-loop system (17) by

$$P_c(\mu) = \det(\mu^2\mathbf{M} + \mu\hat{\mathbf{C}} + \hat{\mathbf{K}}). \tag{18}$$

Since $\hat{\mathbf{C}}$ and $\hat{\mathbf{K}}$ are in general neither symmetric nor positive semi-definite, with poorly chosen $\mathbf{f}$ and $\mathbf{g}$, the closed-loop system may be unstable. The closed-loop eigenvalue problem (17) has $2n$ eigenvalues μ_i and eigenvectors $\mathbf{y}_i$. Throughout this paper, we will assume that the sets $\{\lambda_i\}_{i=1}^{2n}$ and $\{\mu_i\}_{i=1}^{2n}$ are closed under conjugation, although all our results hold without alteration for generally complex spectral data. The assumption of self-conjugacy ensures a real control function $u(t)$ which is realizable.

The single-input pole assignment problem can be stated as follows.

Problem 1. *Given* $\mathbf{M}, \mathbf{C}, \mathbf{K}, \mathbf{b}$ *and a self-conjugate set* $\{\mu_i\}_{i=1}^{2n}$, *find* $\mathbf{f}$ *and* $\mathbf{g}$ *which assign the poles of equation* (14) *to the prescribed set* $\{\mu_i\}_{i=1}^{2n}$.

It is well known that if the system (14) is controllable in the sense that the λ_i are distinct and $\mathbf{b}^{\mathrm{T}}\mathbf{x}_i \neq 0$ for $i = 1, 2, \ldots, 2n$, then the single-input pole assignment problem is solvable, and moreover, the solution is unique. We may thus choose unique real vectors $\mathbf{f}$ and $\mathbf{g}$ and assign the poles of equation (14) to an arbitrarily chosen self-conjugate set $\{\mu_i\}_{i=1}^{2n}$. The traditional approach to actually determine the solution is to write the system (12), (13) in its first order realization form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\mathbf{z}\\ \dot{\mathbf{z}}\end{pmatrix} = \begin{bmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C}\end{bmatrix}\begin{pmatrix}\mathbf{z}\\ \dot{\mathbf{z}}\end{pmatrix} + \begin{pmatrix}\mathbf{0}\\ \mathbf{M}^{-1}\mathbf{b}\end{pmatrix}(\mathbf{g}^{\mathrm{T}}\ \ \mathbf{f}^{\mathrm{T}})\begin{pmatrix}\mathbf{z}\\ \dot{\mathbf{z}}\end{pmatrix} \tag{19}$$

and to apply one of the well-known algorithms, e.g. reference [2], for pole assignment of a first order system.

Although the pole assignment problem by single-input control and its solution seem satisfactory from a theoretical standpoint, one may still encounter some difficulties when implementing this methodology in practice. Among other difficulties we may find that the unique solution obtained by the single-input control is sensitive to perturbations in the system parameters or the applied control Hence, the poles may be shifted incorrectly due to model uncertainties and simplified assumptions arising from practical design problems. It may also happen that the control force $u(t)$ required to relocate the poles is so large that it cannot be implemented in practice without causing a rapid structural fatigue.

We may overcome the practical difficulties mentioned above by using multi-input control forces where the closed-loop system (12), (13) is replaced by

$$\mathbf{M}\ddot{\mathbf{z}} + \mathbf{C}\dot{\mathbf{z}} + \mathbf{K}\mathbf{z} = \mathbf{B}\mathbf{u}(t), \quad \mathbf{B} \in \Re^{n\times m}, \tag{20}$$

with the control vector

$$\mathbf{u}(t) = \mathbf{F}^{\mathrm{T}}\dot{\mathbf{z}}(t) + \mathbf{G}^{\mathrm{T}}\mathbf{z}(t), \quad \mathbf{F}, \mathbf{G} \in \Re^{n\times m}. \tag{21}$$

Equations (20) and (21) may thus be written as

$$\mathbf{M}\ddot{\mathbf{z}} + \tilde{\mathbf{C}}\dot{\mathbf{z}} + \tilde{\mathbf{K}}\mathbf{z} = \mathbf{0}, \tag{22}$$

where

$$\tilde{\mathbf{C}} = \mathbf{C} - \mathbf{B}\mathbf{F}^{\mathrm{T}}, \quad \tilde{\mathbf{K}} = \mathbf{K} - \mathbf{B}\mathbf{G}^{\mathrm{T}}. \tag{23}$$

The separation of variables (16) now leads to the eigenvalue problem

$$(\mu^2\mathbf{M} + \mu\tilde{\mathbf{C}} + \tilde{\mathbf{K}})\mathbf{y} = \mathbf{0}. \tag{24}$$

with $2n$ eigenvalues μ_i and their $2n$ associated eigenvectors $\mathbf{y}_i$. The multi-input pole assignment problem is defined as follows:

Problem 2. *Given* $\mathbf{M}, \mathbf{C}, \mathbf{K}, \mathbf{B}$ *and a self-conjugate set* $\{\mu_i\}_{i=1}^{2n}$, *find* $\mathbf{F}$ *and* $\mathbf{G}$ *which assign the poles of system* (20), (21) *to the prescribed set* $\{\mu_i\}_{i=1}^{2n}$.

Partitioning

$$\mathbf{B} = [\mathbf{B}_1|\mathbf{b}_m], \quad \mathbf{F} = [\mathbf{F}_1|\mathbf{f}_m], \quad \mathbf{G} = [\mathbf{G}_1|\mathbf{g}_m], \qquad \mathbf{B}_1, \mathbf{F}_1, \mathbf{G}_1 \in \Re^{n\times(m-1)}$$

and denoting

$$\breve{\mathbf{C}} = \mathbf{C} - \mathbf{B}_1\mathbf{F}_1^{\mathrm{T}}, \quad \breve{\mathbf{K}} = \mathbf{K} - \mathbf{B}_1\mathbf{G}_1^{\mathrm{T}}, \breve{u}(t) = \mathbf{g}_m^{\mathrm{T}}\dot{\mathbf{z}} + \mathbf{f}_m^{\mathrm{T}}\mathbf{z}, \tag{25}$$

we may write the multi-input controlled system (20), (21) as a single-input controlled system

$$\mathbf{M}\ddot{\mathbf{z}} + \breve{\mathbf{C}}\dot{\mathbf{z}} + \breve{\mathbf{K}}\mathbf{z} = \mathbf{b}_m\breve{u}(t), \tag{26}$$

with $\mathbf{F}_1$ and $\mathbf{G}_1$ arbitrarily chosen. It may thus be deduced that the multi-input pole assignment problem has a continuous family of solutions depending on $2n(m-1)$ arbitrary parameters (the elements of $\mathbf{F}_1$ and $\mathbf{G}_1$), provided that certain controllability requirements are satisfied.

This arbitrariness in selecting the control has been used by Kautsky *et al.* [3] to determine a solution to the pole assignment problem which is in some sense robust and insensitive to perturbations in the model parameters. This result has been generalized by Chu and Datta [4] to second order systems as equation (1). The objective of this paper is to address the problem of reducing the control forces required to relocate the poles by using multi-input control. In fact, we address the more general problem of *partial pole assignment problem* where only $2p \leqslant 2n$ poles of the system are assigned while leaving the other poles unchanged. We develop algorithms for partial pole assignment using multi-input control. It will be explained why a significant reduction in the control forces can be achieved and the results will be demonstrated by simple examples. The analysis is carried out in the natural second order form of vibrating systems, without using a first order realization that destroys the symmetry and definiteness of the matrices involved.

The paper is organized as follows. In section 2 we briefly summarize the recently derived solution by Datta *et al.* [5] to the partial pole assignment problem by single-input control. We then consider the partial assignment of poles by multi-input control in Section 3. Two methods are developed, one associated with a multi-step solution, the other by a single step. Numerical examples demonstrating significant reductions in the magnitude of the control forces by using multi-input control are presented in Section 4. Conclusions are finally drawn in Section 5.

2. SINGLE-INPUT CONTROL

In order to fully comprehend the underlying idea and mechanism of reducing the control forces, it is useful to consider first the recently developed explicit solution [5] to the partial pole assignment problem of a second order system using single-input control.

Let $\{\lambda_i\}_{i=1}^{2p}$, $p \leqslant n$, be a self-conjugate set. Let

$$\Lambda_1 = \mathrm{diag}\{\lambda_1, \lambda_2, \dots, \lambda_{2p}\}, \mathbf{X}_1 = [\mathbf{x}_1|\mathbf{x}_2| \dots \mathbf{x}_{2p}]. \tag{27}$$

where λ_i and $\mathbf{x}_i$, $i = 1, 2, \dots, 2p$, belong to subset of poles, and the associated eigenvectors, of the open-loop system (1) respectively. We have given an explicit solution to the following *single-input partial pole assignment problem.*

Problem 3. Given $\mathbf{M}$, $\mathbf{K}$, Λ_1, $\mathbf{X}_1$, $\mathbf{b}$ *and a self-conjugate set* $\{\alpha_i\}_{i=1}^{2p}$, *find* $\mathbf{f}$ *and* $\mathbf{g}$ *which assign the poles of system* (12), (13) *to the set*

$$\mu_i = \begin{cases} \alpha_i, & i = 1, 2, \dots, 2p, \\ \lambda_i, & i = 2p + 1, \dots, 2n, \end{cases} \tag{28}$$

where λ_i *are poles of the open-loop system* (1).

The solution to Problem 3 is given by

$$\mathbf{f} = \mathbf{M}\mathbf{X}_1\Lambda_1\mathbf{h}, \qquad \mathbf{g} = -\mathbf{K}\mathbf{X}_1\mathbf{h}, \tag{29}$$

where $\mathbf{h} = (h_1, h_2, \dots, h_{2p})^{\mathrm{T}}$ is defined by its components

$$h_j = \frac{1}{\mathbf{b}^{\mathrm{T}}\mathbf{x}_j}\frac{\mu_j - \lambda_j}{\lambda_j}\prod_{\substack{i=1 \\ i \neq j}}^{2p}\frac{\mu_i - \lambda_i}{\lambda_i - \lambda_j}, \quad j = 1, 2, \dots, 2p. \tag{30}$$

An important point to observe is that if we relocate the poles to only $1/\beta$, $\beta > 1$, of the distance $\lambda_j - \alpha_i$, $i = 1, 2, \dots, 2p$, then the magnitude of h_j is reduced by a factor of β^{-2p}. A reduction in $\mathbf{f}$ and $\mathbf{g}$ is correspondingly expected via equation (29). This is one of the mechanisms which we will use to reduce the control force applied by the multi-input control.

3. MULTI-INPUT CONTROL

We define the multi-input partial pole assignment problem as follows:

Problem 4. *Given* $\mathbf{M}, \mathbf{K}, \Lambda_1, \mathbf{X}_1, \mathbf{B}$ *and a self-conjugate set* $\{\alpha_i\}_{i=1}^{2p}$, *find* $\mathbf{F}$ *and* $\mathbf{G}$ *which assign the poles of system* (20), (21) *to the set*

$$\mu_i = \begin{cases} \alpha_i, & i = 1, 2, \ldots, 2p, \\ \lambda_i, & i = 2p+1, 2p+2, \ldots, 2n. \end{cases} \tag{31}$$

We present two different solutions to this problem, which we call the *multi* and *single-step* methods. As stated in the introduction, Problem 4 has a continuous family of solutions. One of our objectives will be, if possible, to select from this family of solutions, those with a norm which is small in some sense.

3.1. MULTI-STEP SOLUTION

The equations governing the dynamics of the multi-input closed-loop system (20), (21) may be written equivalently in the form

$$\mathbf{M}\ddot{\mathbf{z}} + \mathbf{C}\dot{\mathbf{z}} + \mathbf{K}\mathbf{z} = \sum_{i=1}^{m} \mathbf{b}_i(\mathbf{f}_i^{\mathrm{T}}\dot{\mathbf{z}} + \mathbf{g}_i^{\mathrm{T}}\mathbf{z}), \tag{32}$$

where $\mathbf{b}_i$, $\mathbf{f}_i$ and $\mathbf{g}_i$ are the ith columns of $\mathbf{B}$, $\mathbf{F}$ and $\mathbf{G}$ respectively. We may regard this equation as m successive assignments of poles by single-input control. For the sake of definiteness, consider a segment line $\mathfrak{I}_j$ in the complex plane with ends at λ_j and μ_j. Let $\mathfrak{I}_j$ be divided into m equal intervals. Let the end of the kth interval (closer to μ_j) be ξ_{jk}, as shown in Figure 1. Then in the first step we may determine the first column of $\mathbf{F}$ and $\mathbf{G}$ by solving Problem 3 with the substitution $\mathbf{b}_1 \to \mathbf{b}$, $\mathbf{f}_1 \to \mathbf{f}$, $\mathbf{g}_1 \to \mathbf{g}$ and $\{\xi_{j1}\}_{j=1}^{2p} \to \{\mu_j\}_{j=1}^{2p}$. This problem can be solved by equations (29)

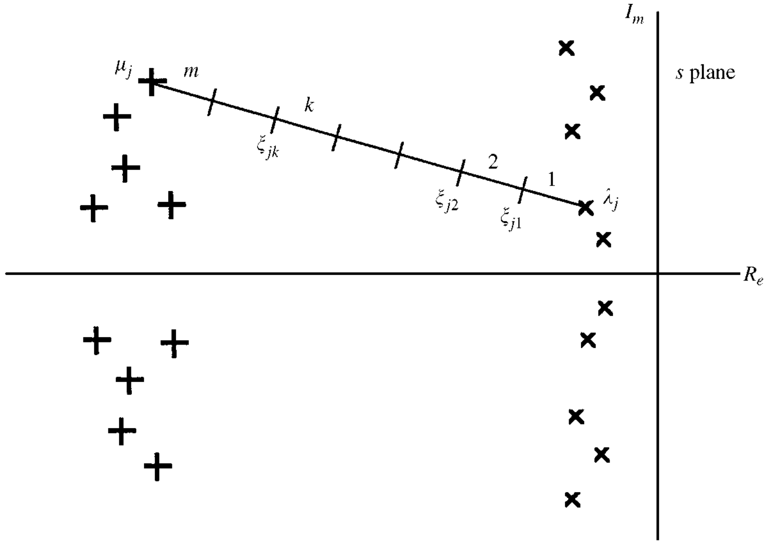

Figure 1. Multi-step assignment: ×, an open-loop system's pole; +, a closed loop system's pole.

and (30). In the second stage, once $\mathbf{f}_1$ and $\mathbf{g}_1$ are known, we at least in principle may determine $\mathbf{f}_2$ and $\mathbf{g}_2$ that assign the eigenvalues $\{\xi_{j1}\}_{j=1}^{2p}$ of the system

$$\mathbf{M}\ddot{\mathbf{x}} + (\mathbf{C} - \mathbf{b}_1\mathbf{f}_1^{\mathrm{T}})\dot{\mathbf{x}} + (\mathbf{K} - \mathbf{b}_1\mathbf{g}_1^{\mathrm{T}})\mathbf{x} = \mathbf{0} \tag{33}$$

to the set $\{\xi_{j2}\}_{j=1}^{2p}$ by using a single-input control

$$\mathbf{M}\ddot{\mathbf{z}} + (\mathbf{C} - \mathbf{b}_1\mathbf{f}_1^{\mathrm{T}})\dot{\mathbf{z}} + (\mathbf{K} - \mathbf{b}_1\mathbf{g}_1^{\mathrm{T}})\mathbf{z} = \mathbf{b}_2(\mathbf{f}_2^{\mathrm{T}}\dot{\mathbf{z}} + \mathbf{g}_2^{\mathrm{T}}\mathbf{z}). \tag{34}$$

This problem *cannot* be solved by equations (29) and (30) since $\mathbf{C} - \mathbf{b}_1\mathbf{f}_1^{\mathrm{T}}$ and $\mathbf{K} - \mathbf{b}_1\mathbf{g}_1^{\mathrm{T}}$ are generally non-symmetric matrices. An effective method to tackle such problem will be developed later in this section. Let $\mathbf{C}_1 = \mathbf{C}$, $\mathbf{K}_1 = \mathbf{K}$ and

$$\mathbf{C}_k = \mathbf{C} - \sum_{i=1}^{k-1} \mathbf{b}_i\mathbf{f}_i^{\mathrm{T}}, \quad \mathbf{K}_k = \mathbf{K} - \sum_{i=1}^{k-1} \mathbf{b}_i\mathbf{g}_i^{\mathrm{T}}, \; k = 2, 3, \ldots, m. \tag{35}$$

Then continuing in the manner described above, we may determine in the kth step the control vectors $\mathbf{f}_k$ and $\mathbf{g}_k$ which assign the eigenvalues $\{\xi_{j,k-1}\}_{j=1}^{2p}$ of

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}_k\dot{\mathbf{x}} + \mathbf{K}_k\mathbf{x} = 0 \tag{36}$$

to the set $\{\xi_{jk}\}_{j=1}^{2p}$ by using the single-input feedback control

$$\mathbf{M}\ddot{\mathbf{z}} + \mathbf{C}_k\dot{\mathbf{z}} + \mathbf{K}_k\mathbf{z} = \mathbf{b}_k(\mathbf{f}_k^{\mathrm{T}}\dot{\mathbf{z}} + \mathbf{g}_k^{\mathrm{T}}\mathbf{z}) \tag{37}$$

until the complete matrices $\mathbf{F}$ and $\mathbf{G}$ are determined.

We now develop an algorithm for pole assignment in the kth stage. Note first that the eigenvalue problem associated with equation (37) is

$$(\xi^2\mathbf{M} + \xi(\mathbf{C}_k - \mathbf{b}_k\mathbf{f}_k^{\mathrm{T}}) + (\mathbf{K}_k - \mathbf{b}_k\mathbf{g}_k^{\mathrm{T}}))\,\mathbf{y} = \mathbf{0}. \tag{38}$$

Theorem 1. *For $j = 1, 2, \ldots, k$, $k \leqslant m$, let $\mathbf{f}_j$ and $\mathbf{g}_j$ be chosen*, respectively, as

$$\mathbf{f}_j = \mathbf{M}\mathbf{X}_1\Lambda_1\mathbf{h}_j, \quad \mathbf{g}_j = -\mathbf{K}\mathbf{X}_1\mathbf{h}_j, \tag{39}$$

with arbitrary vectors $\mathbf{h}_j$ of appropriate dimension. Then $\{\lambda_i, \mathbf{x}_i\}_{i=2p+1}^{2n}$ are eigenpairs of the closed-loop system (37).

Proof. Let

$$\mathbf{P}_k = \sum_{i=1}^{k} \mathbf{b}_i\mathbf{h}_i^{\mathrm{T}}. \tag{40}$$

Then, using equation (39), we may write equation (38) in the form

$$(\xi^2\mathbf{M} + \xi(\mathbf{C} - \mathbf{P}_k\Lambda_1\mathbf{X}_1^{\mathrm{T}}\mathbf{M}) + \mathbf{K} + \mathbf{P}_k\mathbf{X}_1^{\mathrm{T}}\mathbf{K})\,\mathbf{y} = \mathbf{0}. \tag{41}$$

Substituting $\lambda_r \to \xi$, $\mathbf{x}_r \to \mathbf{y}$ for $r > 2p$ we have

$$(\lambda_r^2\mathbf{M} + \lambda_r(\mathbf{C} - \mathbf{P}_k\Lambda_1\mathbf{X}_1^{\mathrm{T}}\mathbf{M}) + \mathbf{K} + \mathbf{P}_k\mathbf{X}_1^{\mathrm{T}}\mathbf{K})\,\mathbf{x}_r = \mathbf{P}_k(-\lambda_r\Lambda_1\mathbf{X}_1^{\mathrm{T}}\mathbf{M} + \mathbf{X}_1^{\mathrm{T}}\mathbf{K})\,\mathbf{x}_r = \mathbf{0} \tag{42}$$

by virtue of equation (3) and the first orthogonal relation for the symmetric definite quadratic pencil obtained in reference [5] (see equation (20) of that paper). □

It will now be shown how to determine $\mathbf{h}_k$. We substitute $\xi_{jk} \to \xi$ and $\mathbf{y}_j \to \mathbf{y}$, $j = 1, 2, \ldots, 2p$, in equation (38):

$$(\xi_{jk}^2 \mathbf{M} + \xi_{jk} \mathbf{C}_k + \mathbf{K}_k)\mathbf{y}_j = \mathbf{b}_k(\xi_{jk} \mathbf{f}_k^{\mathrm{T}} + \mathbf{g}_k^{\mathrm{T}})\mathbf{y}_j. \tag{43}$$

Note that $(\xi_{jk} \mathbf{f}_k^{\mathrm{T}} + \mathbf{g}_k^{\mathrm{T}})\mathbf{y}_j$ is a scalar quantity. Hence, denoting

$$\tau_{jk} = (\xi_{jk} \mathbf{f}_k^{\mathrm{T}} + \mathbf{g}_k^{\mathrm{T}})\mathbf{y}_j, \quad j = 1, 2, \ldots, 2p, \tag{44}$$

equation (43) can be written as

$$(\xi_{jk}^2 \mathbf{M} + \xi_{jk} \mathbf{C}_k + \mathbf{K}_k)\hat{\mathbf{y}}_j = \mathbf{b}_k, \quad j = 1, 2, \ldots, 2p, \tag{45}$$

where $\hat{\mathbf{y}}_j = \tau_{jk}^{-1} \mathbf{y}_j$. Since $\mathbf{C}_k$ and $\mathbf{K}_k$ are explicitly known at the start of the kth step we may solve equation (45) and evaluate $\hat{\mathbf{y}}_j$. Then using equations (44) and (39) we have

$$\tau_{jk} = \mathbf{h}_k^{\mathrm{T}}(\xi_{jk} \Lambda_1 \mathbf{X}_1^{\mathrm{T}} \mathbf{M} - \mathbf{X}_1^{\mathrm{T}} \mathbf{K})\mathbf{y}_j \tag{46}$$

or

$$\hat{\mathbf{y}}_j^{\mathrm{T}}(\xi_{jk} \mathbf{M} \mathbf{X}_1 \Lambda_1 - \mathbf{K} \mathbf{X}_1)\mathbf{h}_k = \mathbf{1}, \quad j = 1, 2, \ldots, 2p. \tag{47}$$

Define

$$\hat{\mathbf{Y}} = [\hat{\mathbf{y}}_1 | \hat{\mathbf{y}}_2 | \ldots | \hat{\mathbf{y}}_{2p}] \tag{48}$$

and

$$\Psi_k = \operatorname{diag}\{\xi_{1k}, \xi_{2k}, \ldots, \xi_{2p,k}\}. \tag{49}$$

Then the $2p$ linear equations (47) can be written in matrix form as

$$\mathbf{T}_k \mathbf{h}_k = s, \tag{50}$$

where

$$\mathbf{T}_k = \hat{\mathbf{Y}}^{\mathrm{T}}(\Psi_k \mathbf{M} \mathbf{X}_1 \Lambda_1 - \mathbf{K} \mathbf{X}_1) \tag{51}$$

and

$$\mathbf{s} = (1, 1, \ldots, 1)^{\mathrm{T}} \in \Re^{2p}. \tag{52}$$

Hence $\mathbf{h}_k$ is determined by equation (50) and $\mathbf{f}_k$ and $\mathbf{g}_k$ are determined by equation (39). We summarize the multi-input partial pole assignment method in the following algorithm.

Algorithm 1. *Multi-input pole assignment for damped vibrating system.*

Input: Three symmetric matrices $\mathbf{M}, \mathbf{C}, \mathbf{K} \in \Re^{n \times n}$, $\mathbf{M}$ positive definite; $\mathbf{B} \in \Re^{n \times m}$, $1 \leqslant m \leqslant n$; and a self-conjugate set $\{\mu_j\}_{j=1}^{2p}$, $1 \leqslant p \leqslant n$.

Procedure:

(1) Calculate (or measure by modal test) $2p$ self-conjugate eigenpairs $\{\lambda_i, \mathbf{x}_i\}_{i=1}^{2p}$ of the open-loop eigenvalue problem (3). Define Λ_1, $\mathbf{X}_1$, as in equation (27), and $\{\xi_{j0}\}_{j=1}^{2p} = \{\xi_j\}_{j=1}^{2p}$.

(2) Determine the intermediate poles $\{\xi_{jk}\}_{j=1}^{2p}$, to be assigned stepwise at the kth step, $k = 1, 2, \ldots, m-1$, by dividing the lines connecting λ_j and μ_j to m equal intervals (as in Figure 1). Set $\{\xi_{jm}\}_{j=1}^{2p} = \{\mu_j\}_{j=1}^{2p}$.

(3) Set $\mathbf{s} = (1, 1, \ldots, 1)^{\mathrm{T}} \in \Re^{2p}$.

(4) For $k = 1, 2, \ldots, m$

  (4.1) For $j = 1, 2, \ldots, 2p$

    (4.1.1) Solve the linear system (45) for $\hat{\mathbf{y}}_j$.

  (4.2) Define $\hat{\mathbf{Y}}$ and Ψ_k for the kth step by using equations (48) and (49).

  (4.3) Using equation (51) solve equation (50) for $\mathbf{h}_k$.

  (4.5) Evaluate $\mathbf{f}_k$ and $\mathbf{g}_k$ from equation (39).

Output: Two control matrices $\mathbf{F} = [\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_m]$ and $\mathbf{G} = [\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_m]$. With these matrices, the poles of the closed-loop system (20), (21) are the prescribed values $\{\mu_i\}_{j=1}^{2p}$ and $\{\lambda_i\}_{i=2p+1}^{2n}$. Moreover, let $\mathbf{F}_k = [\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_k]$, $\mathbf{G}_k = [\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_k]$ and $\mathbf{B}_k = [\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k]$; then the poles of

$$\mathbf{M}\ddot{\mathbf{z}} + \mathbf{C}\dot{\mathbf{z}} + \mathbf{K}\mathbf{z} = \mathbf{B}_k(\mathbf{F}_k^{\mathrm{T}}\dot{\mathbf{z}} + \mathbf{G}_k^{\mathrm{T}}\mathbf{z})$$

are $\left\{\{\xi_{jk}\}_{j=1}^{2p}, \{\lambda_i\}_{i=2p+1}^{2n}\right\}$ for $k = 1, 2, \ldots, m$.

Remark. The output in Algorithm 1 does depend on the order of poles $\{\lambda_i\}_{i=1}^{2n}$ and $\{\mu_i\}_{i=1}^{2n}$. Since the sets of the poles of the open- and closed-loop systems can be numbered in any order, we may renumber the poles $\{\mu_i\}_{i=1}^{2n}$ and obtain generally different control $\mathbf{F}$ and $\mathbf{G}$ from the algorithm. This provides us with some freedom in the design of the controller.

3.2. SINGLE-STEP SOLUTION

We now show that the partial pole assignment can be achieved in a single-step process in which the family of solution is characterized by an arbitrarily chosen matrix.

Theorem 1 may be recast in the following matrix form.

Theorem 2. *Let*

$$\mathbf{F} = \mathbf{M}\mathbf{X}_1\Lambda_1\mathbf{H}, \mathbf{G} = -\mathbf{K}\mathbf{X}_1\mathbf{H}, \mathbf{H} \in \Re^{2p\times m}. \tag{53}$$

Then, for any choice of $\mathbf{H}$, *the pairs* $\{\lambda_i, \mathbf{x}_i\}_{i=2p+1}^{2n}$ *are eigenpairs of the closed-loop system* (20), (21).

In order to use Theorem 2 to solve the multi-input partial role assignment problem, we need to choose $\mathbf{H}$ which will move $\{\lambda_j\}_{j=1}^{2p}$ of the open-loop system (3) to $\{\mu_j\}_{j=1}^{2p}$ in the closed-loop system, if that is possible. If there is such an **H**, then there exist an eigenvector matrix $\mathbf{Y} \in C^{n \times 2p}$,

$$\mathbf{Y} = [\mathbf{y}_1 | \mathbf{y}_2 | \ldots | \mathbf{y}_{2p}] \tag{54}$$

and

$$\mathbf{D} = \text{diag}\{\mu_1, \mu_2, \ldots, \mu_{2p}\}, \tag{55}$$

which are such that

$$\mathbf{MYD}^2 + (\mathbf{C} - \mathbf{BF}^{\mathrm{T}})\mathbf{YD} + (\mathbf{K} - \mathbf{BG}^{\mathrm{T}})\mathbf{Y} = \mathbf{0}. \tag{56}$$

Substituting for **F**, **G** and rearranging, we have

$$\begin{aligned} \mathbf{MYD}^2 + \mathbf{CYD} + \mathbf{KY} &= \mathbf{BH}^{\mathrm{T}}(\Lambda_1 \mathbf{X}_1^{\mathrm{T}} \mathbf{MYD} - \mathbf{X}_1^{\mathrm{T}} \mathbf{KY}) \\ &= \mathbf{BH}^{\mathrm{T}} \mathbf{U}^{\mathrm{T}} \\ &= \mathbf{BV}^{\mathrm{T}} \end{aligned} \tag{57}$$

with the obvious definition of **U** and

$$\mathbf{V} = \mathbf{UH} \tag{58}$$

is a matrix that will depend on the scaling chosen for the eigenvectors in **Y**. Let $\hat{\mathbf{E}}$ be defined by the $2p \times m$ matrix

$$\hat{\mathbf{E}} = [\hat{e}_{ij}] = \begin{cases} 1, & j = i + km, k = 0, 1, \ldots, \\ 0, & \text{elsewhere.} \end{cases} \tag{59}$$

Now, any matrix **H** which leads to $\mathbf{V} = \hat{\mathbf{E}}$ will suffice, so one solution can be obtained as follows.

To begin we construct the matrix **Y** by solving for each of the eigenvectors $\mathbf{y}_i$ using the equations

$$(\mu_j^2 \mathbf{M} + \mu_j \mathbf{C} + \mathbf{K})\mathbf{y}_j = \mathbf{B}\hat{\mathbf{e}}_j, \tag{60}$$

where $\hat{\mathbf{e}}_j^{\mathrm{T}}$ is the jth row of $\hat{\mathbf{E}}$. Next, we solve the system $\mathbf{UH} = \hat{\mathbf{E}}$ for **H** and hence determine the corresponding **F** and **G**. Note that **U** is invertible whenever the system is controllable.

For any $m \times m$ invertible matrix **W** we may obtain another solution by replacing **B** by **BW** and **F** and $\mathbf{FW}^{-\mathrm{T}}$ throughout.

4. EXAMPLES

To demonstrate the effectiveness of methods of the previous sections we give here the results of some simple comparisons between the control forces that are required to reassign some poles of a system.

All calculations were done in IEEE standard double-precision arithmetic (machine accuracy of about 2×10^{-16}) on the linear algebra package Matlab. All numbers quoted are correctly rounded to the number of figures shown.

Example 1. In this examples $m = 3$, $p = 2$ and n varies between 10 and 40. For the multi-input control $\mathbf{B}$ is an $n \times m$ identity, i.e.

$$\mathbf{B} = \begin{bmatrix} \mathbf{I}_m \\ \mathbf{0} \end{bmatrix}.$$

In the single-input control each element of $\mathbf{b}$ is the sum of the elements in the corresponding row of $\mathbf{B}$. The other given matrices are $\mathbf{M} = \mathbf{I}$, $\mathbf{C} = \mathbf{0}$ and

$$\mathbf{K} = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -2 & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix}.$$

The $2p = 4$ eigenvalues with smallest absolute values are reassigned to the values $\lambda_{2k-1} = -k + \sqrt{-10k}$, $\lambda_{2k} = \bar{\lambda}_{2k-1}$, $k = 1, 2$. For multi-step procedure, this reassignment is done in $m = 3$ steps of equal length.

The control matrices $\mathbf{F}$ and $\mathbf{G}$ for the multi- and single-step methods and the control vectors $\mathbf{f}$ and $\mathbf{g}$ for the same $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ in the single-input control have been found. The 2-norm of the control vectors and matrices are shown in Table 1. We can see that, as expected, the task of reassigning four poles of the system while keeping the other poles unchanged is achieved with a significantly reduced control effort when the multi-input control strategies proposed in the paper are used.

Example 2. In this example we repeat Example 1, but here $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ have tridiagonal symmetric structure with elements that are uniformly distributed

TABLE 1

Norms of control vectors and matrices for Example 1

	Single-input control		Multi-input single step		Multi-input multi-step	
n	$\lvert f \rvert$	$\lvert g \rvert$	$\lvert F \rvert$	$\lvert G \rvert$	$\lvert F \rvert$	$\lvert G \rvert$
10	96433	279925	5057	70907	20685	20212
20	86137	251060	3978	53262	16652	16575
40	644593	1877483	27010	396778	118597	120908

TABLE 2

Norms of control vectors and matrices for Example 2

	Single-input control		Multi-input single step		Multi-input multi-step	
n	$\|f\|$	$\|g\|$	$\|F\|$	$\|G\|$	$\|F\|$	$\|G\|$
10	$3{\cdot}18\times10^{6}$	$2{\cdot}94\times10^{6}$	$1{\cdot}49\times10^{4}$	$1{\cdot}02\times10^{4}$	$4{\cdot}38\times10^{4}$	$4{\cdot}15\times10^{4}$
20	$1{\cdot}36\times10^{7}$	$1{\cdot}14\times10^{7}$	$8{\cdot}93\times10^{4}$	$7{\cdot}46\times10^{4}$	$1{\cdot}03\times10^{6}$	$8{\cdot}64\times10^{5}$
40	$4{\cdot}54\times10^{15}$	$6{\cdot}02\times10^{15}$	$6{\cdot}64\times10^{10}$	$3{\cdot}43\times10^{10}$	$2{\cdot}51\times10^{10}$	$3{\cdot}43\times10^{10}$

random numbers in [0, 1]. The results presented in Table 2 for three cases are typical of those we achieved over a large number of trials and give the flavour of what one might expect from the methods of the paper, i.e. reduction in the control effort by using the multi-input control.

While the solution via single-input control is unique, the formulation via multi-input single step expresses a continuous family of possible solutions in terms of an arbitrary matrix **H**. A search within the family of solutions gives a design freedom that may reduce the control effort. It is evident from Table 2 that such a solution may be better in certain situations than that obtained by the multi-input multi-step solution.

5. CONCLUSIONS

We have suggested in this paper that the multi-input partial pole assignment problem can be considered as a sequence of pole assignments by single-input control. The poles are pushed gradually from their initial position to the final destination by the applied control. It is clear from the explicit solution to the single-input case obtained by Datta *et al.* [5] that such an approach reduces the control effort significantly. Numerical examples confirm this observation. The algorithms presented describe a closed-form, non-iterative, solution to the problem. The entire analysis is carried out by using the natural set of second order differential equations that governs the motion of vibratory systems, and no first order realization is used.

Investigations of the related problem of assigning both poles and zeros of a system, and pole assignment for a distributed parameter system can be found in references [6, 7].

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